Norm inequality and partition of (pure) states in $C(M) \otimes M_n(\mathbb{C})$

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Abstract

Given a spectral triple, a *noncommutative distance* between states can be defined. Even for "compact" noncommutative spaces, infinite distances between states can arise. We first study this phenomenon in a general setting and then specialise to certain spectral triples over $C(M)\otimes M_n(\mathbb{C})$. The "bounded components" induce a partition of the space of states, which we describe in our particular case.

In his article [6], Connes defines the notion of *spectral triple*, a noncommutative version of spin manifolds. A spectral triple is made up of an involutive algebra $\mathscr{A}$ acting faithfully on a Hilbert space $\mathscr{H}$, together with an unbounded self-adjoint operator D on $\mathscr{H}$. We denote by $\| \, \|$ the norm on $\mathscr{A}$ induced by its representation on $\mathscr{H}$. The triple $\mathscr{A}, \mathscr{H}$ and D must fulfill certain conditions to form a spectral triple. In particular, the commutator $[D, a]$ must be bounded for any $a \in \mathscr{A}$. The algebra $\mathscr{A}$ can be commutative or not, hence the name "noncommutative geometry".

In the commutative case, the conditions imply that $\mathscr{A} = C^\infty(M)$, D is the Dirac operator on the spin manifold M and $\mathscr{H}$ is the set of L^2 spinors on M (see [7]). In the following, we will use a "simplified" version of spectral triples, and we will only use the conditions (see section 2 of [7]):

- D is an unbounded self-adjoint operator.
- $\mathscr{A} = \{T \in \mathfrak{A} \colon T \text{ and } [D,T] \in \bigcap_{m>0} \operatorname{Dom} \delta^m\}$ where $\mathfrak{A}$ is a von Neumann algebra represented on $\mathscr{H}$, $\delta(T) = [|D|, T]$ and $\operatorname{Dom}\delta$ is the set of all operators T such that $T(\operatorname{Dom}|D|) \subseteq \operatorname{Dom}|D|$ and $\delta(T)$ extends to a bounded linear map on $\mathscr{H}$.

One can prove (see [8], lemma 2.1) that $\mathscr{A}$ is an involutive algebra. We call $A = \overline{\mathscr{A}}$ the closure in $B(\mathscr{H})$ of $\mathscr{A}$ in the operator norm. This ensures that the bounded positive linear forms σ on $\mathscr{A}$ which satisfy $\sigma(1) = 1$ extend to *states* of the C^*-algebra A. We will call these forms the *states of $\mathscr{A}$*.

On a spectral triple, there are no "points", they are replaced by the (pure) states of $\mathscr{A}$. Indeed, a consequence of the Gelfand-Naimark theorem is that the pure states of $C^\infty(M)$ equipped with the sup-norm are the points of M.

Using D, a distance between states of $\mathscr{A}$ can be defined by the formula:

$$d(\sigma, \sigma') = \sup\{|\sigma(a) - \sigma'(a)|, \|[D,a]\| \leqslant 1\},$$

where we use the norm in $B(\mathscr{H})$. There are two natural topologies on the space of states: the *metric topology*, defined on states using the distance d, and the *weak $*$-topology* arising from the evaluation of states on elements of $\mathscr{A}$. The relations between these two topologies were investigated in [13, 15, 16].

In a more physical context, distances were explicitly computed for finite-dimensional noncommutative algebras [9], Moyal spaces [3, 4] and almost commutative geometries [10, 12]. In this latter case, the results were related to Carnot-Carathéodory distance [11].

The distance d has the surprising property that even in a "compact" spectral triple, *i.e.* if $\mathscr{A}$ is unital, some states verify $d(\sigma, \sigma') = \infty$. Even though this property has been known for a long time (see [6]), there seems to be no systematic treatment of this phenomenon. The goal of this article is to start such a study. Given a state σ, we can define the *bounded component* $[\sigma]^b$ of this state as:

$$[\sigma]^b = \{\sigma' : d(\sigma, \sigma') < \infty\}.$$

The underlying idea is that bounded components should induce a *singular foliation* of the space of pure states, which could in turn be analysed using, for instance, the results of [1]. While this result is still far off, the present paper provides a first step in this direction.

To analyse bounded components $[\sigma]^b$, we introduce the notion of *classes of restriction* $[\sigma]^r$ and *connected components* $[\sigma]^c$ of the space of states. In a first section, we establish a hierarchy of the three notions and prove that the inclusions can be strict. We moreover give a necessary and sufficient condition under which restriction classes coincide with bounded components. The last two sections of the paper are devoted to the study of matrix-valued smooth functions over a manifold. We prove that for a certain class of spectral triples over this algebra,

$$[\sigma]^c = [\sigma]^b = [\sigma]^r.$$

We conclude this paper by giving an explicit description of bounded components for this class of spectral triples.

In the following, we will always assume that $\mathscr{A}$ is unital.

1 Partitions of States

1.1 Definition and First Properties

Given a spectral triple $(\mathscr{A}, \mathscr{H}, D)$, we call $\mathrm{Comm}(D)$ the set $\{a \in \mathscr{A} : [D, a] = 0\}$. Notice that $\mathrm{Comm}(D)$ is a unital subalgebra of $\mathscr{A}$ and therefore the restriction of a state of σ to $\mathrm{Comm}(D)$ is a state. We introduce 3 sets of states of $\mathscr{A}$:

Definition 1.1 (restriction class)**.** Given a state σ, we define its *restriction class* $[\sigma]^r$ as:

$$[\sigma]^r = \{\sigma' : \sigma'_{|\,\mathrm{Comm}(D)} = \sigma_{|\,\mathrm{Comm}(D)}\}.$$

Definition 1.2 (bounded component)**.** Given a state σ, we define its *bounded component* $[\sigma]^b$ as:

$$[\sigma]^b = \{\sigma' : d(\sigma, \sigma') < \infty\}.$$

The *connected component* $[\sigma]^c$ of a state σ, in the sense of the metric topology, has an obvious meaning.

Proposition 1.3. *For any state σ of $\mathscr{A}$ we have the following inclusions:*

$$[\sigma]^c \subseteq [\sigma]^b \subseteq [\sigma]^r. \tag{1}$$

Proof. Let us first show that $[\sigma]^b \subseteq [\sigma]^r$ by showing that if $\sigma_{|\,\mathrm{Comm}(D)} \neq \sigma'_{|\,\mathrm{Comm}(D)}$ then $d(\sigma, \sigma') = \infty$: if we can find $a_0 \in \mathrm{Comm}(D)$ such that $\sigma(a_0) - \sigma'(a_0) = \varepsilon \neq 0$, then for any $\lambda \in \mathbb{R}$ $\|[D, \lambda a_0]\| \leqslant 1$ and $|\sigma(\lambda a_0) - \sigma'(\lambda a_0)| = |\lambda\varepsilon| \to \infty$.

The connected component of a state is included in the bounded component. Indeed, if $d(\sigma_0, \sigma_0') = \infty$, then $U_{\sigma_0} = \{\sigma : d(\sigma_0, \sigma) < \infty\}$ and $U_{\sigma_0'} = \{\sigma' : d(\sigma_0', \sigma') < \infty\}$ are two open sets in the sense of the metric topology. Since $d(\sigma_0, \sigma_0') = \infty$, $U_{\sigma_0} \cap U_{\sigma_0'} = \emptyset$ and this shows that σ_0 and σ_0' are not in the same connected component. $\square$

For most algebra $\mathscr{A}$, the set $\mathscr{S}(\mathscr{A})$ of all states of $\mathscr{A}$ is much too large to be studied. We therefore restrict our attention to the set $\mathcal{P}(\mathscr{A})$ of *pure* states, *i.e.* the extreme points of $\mathscr{S}(\mathscr{A})$ – see [2], p.105. Henceforth we consider σ a *pure* state and $[\sigma]^r, [\sigma]^b, [\sigma]^c$ as subsets of $\mathcal{P}(\mathscr{A})$.

In general, the inclusions of (1) are strict. To prove that $[\sigma]^c \subsetneq [\sigma]^b$ can happen, it suffices to consider the algebra $\mathscr{A} = \mathbb{C} \oplus \mathbb{C}$ acting naturally on $\mathscr{H} = \mathbb{C} \oplus \mathbb{C}$, endowed with the dirac operator $D = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$. This algebra has only two pure states:

$$\sigma(x \oplus y) = x \qquad\qquad \sigma'(x \oplus y) = y.$$

It is readily checked that the distance $d(\sigma, \sigma') = \frac{1}{|m|}$. This proves that both states belong to the same bounded component. However, they do not belong to the same connected component.

The example of $[\sigma]^b \subsetneq [\sigma]^r$ is more involved. The first example of such phenomenon was given in [3], section 3. For the convenience of the reader, we give a summary of this article.

A spectral triple is defined by taking $A = \widetilde{\mathbb{K}}$, the unitalisation of the compact operators $\mathbb{K}$ acting diagonally on $\mathscr{H} = l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$. We denote by (e_n) a basis of $\mathscr{H}_0 = l^2(\mathbb{N})$ and by $(f_{m,n})$ the associated matrix units. To define a Dirac operator D on $\mathscr{H}$, we first introduce the unbounded operator ∂_0, whose domain are the finite sums of (e_n), and whose definition comes from $\partial_0 e_m = -\sqrt{m+1}\, e_{m+1}$. It is easy to check that if x, y are finite sums of (e_n), then

$$\langle \partial_0 y, x \rangle = \langle y, \overline{\partial} x \rangle,$$

where $\overline{\partial} e_m = \sqrt{m} e_{m-1}$ and $\overline{\partial} e_0 = 0$. This shows that the domain of the adjoint ∂_0^* of ∂_0 contains the finite sums. Consequently, ∂_0^* is densely defined and ∂_0 is closable (see [14], theorem 5.1.5). Letting $\overline{\partial} = \partial_0^*$ and $\partial = \partial_0^{**}$ we obtain a self-adjoint operator D by the formula

$$D = \begin{pmatrix} 0 & \overline{\partial} \\ \partial & 0 \end{pmatrix}.$$

With these definitions, we see that finite rank operators send $\operatorname{Dom} D$ to itself. Therefore, the finite rank operators belong to $\mathscr{A}$ defined in the introduction.

Since A has a canonical representation on $\mathscr{H}_0$, we can define pure states on A using a unit vector $\psi \in \mathscr{H}_0$: it suffices to set $\sigma_\psi(a) = \langle \psi, a\psi \rangle$. In particular, if we take

$$\psi = \frac{1}{\sqrt{K}} \sum_p \frac{1}{(p+1)^{3/4}} e_p,$$

where $K = \sum_p \frac{1}{(p+1)^{3/2}}$, we can define two pure states by $\sigma_0(a) = \langle e_0, ae_0 \rangle$ and $\underline{\sigma}(a) = \langle \psi, a\psi \rangle$.

The article [3] proves that the distance $d(\sigma_0, \overline{\sigma}) = \infty$ (see proposition 3.10) by considering the sequence:

$$a^{(m)} = \frac{1}{\sqrt{2}} \sum_{p=0}^{m} \sum_{k=p}^{m} \frac{1}{\sqrt{k+1}} f_{p,p}.$$

Notice that $a^{(m)} \in \mathscr{A}$ since the sum is finite for any fixed m. Direct computations prove that $\|[D, a^{(m)}]\| \leqslant 1$ and

$$\left|\sigma_0(a^{(m)}) - \sigma_\psi(a^{(m)})\right| = \left|\frac{1}{\sqrt{2}} \sum_{k=0}^{m} \frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{2}K} \sum_{p=0}^{m} \frac{1}{(p+1)^{3/2}} \sum_{k=p}^{m} \frac{1}{\sqrt{k+1}}\right|$$

One can prove (see [4], proposition 6) that this sum diverges, and therefore $d(\sigma_0, \sigma_\psi) = \infty$. However, using the formulas given in the proposition 5 of [4], it is not difficult to check that $\mathrm{Comm}(D) = \mathbb{C}1$. Hence, σ_0 and σ_ψ belong to the same restriction class, yet they are not in the same bounded component.

1.2 Necessary and Sufficient Condition of Equality

Following the reasoning of [15], we now find a necessary and sufficient condition for $[\sigma]^c = [\sigma]^b$. We will do this in two parts. We start with

Lemma 1.4. *If there is a constant $C > 0$ such that :*

$$\|a\|^\sim \leqslant C\|[D, a]\|$$

where $\| \, \|^\sim$ is a (semi-)norm on the quotient space $\mathscr{A} / \mathrm{Comm}(D)$ defined by

$$\|a\|^\sim = \inf_{a_0 \in \mathrm{Comm}(D)} \|a + a_0\|,$$

then two states belong to the same bounded component if and only if they have the same restriction to $\mathrm{Comm}(D)$. *Moreover, the diameter of any bounded component $[\sigma]$ is bounded by $2C$.*

Remark 1.5. In general, $\mathscr{A}$ is *not* a Banach space, *e.g.* $\mathscr{A} = C^\infty(M)$. However, it is natural to ask when $\| \, \|^\sim$ is actually a *norm*. Proposition 1.9 provides a sufficient condition for this property.voir si on ne peut pas faire mieux (Reed & Simon I, p.268)

Proof. Let σ^1, σ^2 be two states of $\mathscr{A}$ such that $(\sigma^1 - \sigma^2)_{|\,\mathrm{Comm}(D)} = 0$. For any $a \in \mathscr{A}$,

$$|(\sigma^1 - \sigma^2)(a)| = |(\sigma^1 - \sigma^2)(a + a_0)| \leqslant 2\|a + a_0\|$$

where $a_0 \in \mathrm{Comm}(D)$. Taking the infimum over a_0, we prove that $\sigma^1 - \sigma^2$ is continuous for $\| \, \|^\sim$. Using the inequality of the hypothesis, we get:

$$|(\sigma^1 - \sigma^2)(a)| \leqslant 2C\|[D, a]\|$$

which shows that $d(\sigma^1, \sigma^2) \leqslant 2C$. □

The above lemma has a converse:

Lemma 1.6. *Assume that:*

(i) Comm(D) *is norm-closed and possess a norm-closed complement in* A*;*

(ii) $d(\sigma^1,\sigma^2)=\infty \iff \exists a_0 \in \operatorname{Comm}(D) : (\sigma^1-\sigma^2)(a_0)\neq 0$, *i.e.* $[\sigma]^b = [\sigma]^r$*;*

(iii) all bounded components have a diameter that is less than $C_0>0$.

We have the following inequality:

$$\|a\|^\sim \leqslant kC_0\|[D,a]\| \tag{2}$$

where $\|\,\|^\sim$ *is the norm of the quotient space* $A/\operatorname{Comm}(D)$ *and* k *is a constant.*

Remark 1.7. Notice that as soon as Comm(D) is finite dimensional, it is complemented.

Proof. Consider the elements $a\in\mathscr{A}$ as functions on the space of states $\mathscr{S}(\mathscr{A})$, and denote by $\|\,\|_\infty$ the sup-norm of these functions. According to [15] p.219, we can find a constant $k'>0$ such that:

$$\|a\|\leqslant k\|a\|_\infty.$$

The assumption (i) together with the discussion of [5] p.19 ensures that we can find a complement M of Comm(D) in A such that the quotient map is an isomorphism. The definition of $\|\,\|^\sim$ entails that for any $a_0\in M$, $\|[a_0]\|^\sim \leqslant \|a_0\|$. Hence, it suffices to consider $a_0\in M$ to get (2).

Let $a_0\in M$, just like [15] p.219, we can find a state μ such that $\|a_0\|\leqslant k|\mu(a_0)|$. Moreover, using the Hahn-Banach theorem, we can define a linear form ν on A which satisfies $\nu_{|\operatorname{Comm}(D)}=\mu_{|\operatorname{Comm}(D)}$, $\nu(a_0)=0$ and $\|\nu\|=1$.

It is now clear that for such a ν, $\nu(1)=1$, and by [2] II.6.2.5, ν is therefore positive.

The assumption (ii) shows that μ and ν belong to the same bounded component. The final assumption now entails

$$\|a_0\|\leqslant k|\mu(a_0)| = k|\mu(a)-\nu(a)|\leqslant kC_0\|[D,a_0]\|,$$

and such an inequality holds for any $a_0\in M$. □

Restriction classes are useful because theorem 1.8 of [15] applies to them, defining their topology:

Proposition 1.8. *Let $\mathcal{L}_1 = \{a \in \mathscr{A} \colon \|[D,a]\| \leqslant 1\}$. If the image of $\mathcal{L}_1$ in the quotient $A/\operatorname{Comm}(D)$ is totally bounded for $\| \|^{\sim}$, then the metric and weak-$*$-topologies agree on the restriction classes.*

Proof. This is just a restatement of the theorem 1.8 of [15]. It applies when we take (in [15], p.217) $A = A$, $\mathcal{L} = \mathscr{A}$, $L(T) = \|[D,T]\|$, $\mathcal{K} = \operatorname{Comm}(D)$ and η to be a (pure) state of $\operatorname{Comm}(D)$. Then the S of [15] is just the restriction class associated to η. $\square$

Notice that if $\mathcal{L}_1$ is totally bounded, then it is bounded *i.e.* there is a constant C such that if $\|[D,a]\| \leqslant 1$, we can find a $a_0 \in \operatorname{Comm}(D)$ such that $\|a + a_0\| \leqslant C$. Thus $\|a\|^{\sim} \leqslant C\|[D,a]\|$ and the hypothesis of lemma 1.4 is fulfilled.

1.3 Study of $\operatorname{Comm}(D)$ and Extremal Restriction Classes

We now describe $\operatorname{Comm}(D)$ when $\mathscr{A}$ is described as regular part, D is selfadjoint and has compact resolvent:

Proposition 1.9. *If the algebra $\mathscr{A}$ is defined by*

$$\mathscr{A} = \left\{ T \in \mathfrak{A} \,\middle|\, T, [D,T] \in \bigcap_{m>0} \operatorname{Dom} \delta^m \right\},$$

where $\mathfrak{A}$ is a von Neumann algebra, $\delta(T) = [|D|, T]$ and D is a selfadjoint operator with compact resolvent, then $\operatorname{Comm}(D)$ *is a (possibly infinite) sum of finite matrices. In particular, it is norm-closed.*

Proof. Since D is a selfadjoint operator with compact resolvent, we can find a sequence of finite dimensional pairwise orthogonal eigenspaces $E_\lambda \subseteq \mathscr{H}$ indexed by the eigenvalues λ such that

$$\mathscr{H} = \bigoplus E_\lambda.$$

An operator $T \in B(\mathscr{H})$ commutes with D if and only if $T(E_\lambda) \subseteq E_\lambda$. In otherwords, $D' = \bigoplus B(E_\lambda)$ and since E_λ has finite dimension, $B(E_\lambda)$ is just a matrix algebra.

Now, we want to take the intersection of the algebra D' with the algebra $\mathfrak{A}$. The intersection is a sub-von Neumann algebra. The result follows from lemma 1.10. $\square$

Lemma 1.10. *Any sub-von Neumann algebra* $\mathfrak{M}$ *of* $\mathfrak{A} = \bigoplus_{k=0}^{\infty} M_{n_k}(\mathbb{C})$, *where for all* k $n_k < \infty$*, is a sum finite dimensional type I factors.*

Proof. The condition on $\mathfrak{A}$ actually means that there are normal morphisms $\phi_k : \mathfrak{A} \to \mathfrak{A}_k = M_{n_k}(\mathbb{C})$ such that $\bigoplus_{k=0}^{\infty} \phi_k : \mathfrak{A} \to \bigoplus_{k=0}^{\infty} \mathfrak{A}_k$ is an isomorphism.

In the particular case of D', these morphisms are in fact $\phi_k(a) = p_k a = a_{|E_{\lambda_k}}$, where p_k is a central projector of $\mathfrak{A}$.

Now set $\mathfrak{R}_0 = \mathfrak{M}$ and define recursively $\mathfrak{R}_{l+1} = \ker \phi_l \cap \mathfrak{R}_l$. Since ϕ_k is normal, for any l

$$\mathfrak{R}_l = \phi_l(\mathfrak{R}_l) \oplus \mathfrak{R}_{l+1}.$$

Consequently,

$$\mathfrak{M} = \bigoplus_{l=0}^{\infty} \phi_l(\mathfrak{R}_l) \oplus \bigcap_{k=0}^{\infty} \mathfrak{R}_k.$$

The definition of $\mathfrak{R}_l$ entails $\mathfrak{R}_{l+1} \subseteq \bigcap_{k=0}^{l} \ker \phi_k$. The hypothesis on $\bigoplus_{k=0}^{\infty} \phi_k$ implies that $\bigcap_{k=0}^{\infty} \ker \phi_k = \{0\}$ and thus

$$\mathfrak{M} = \bigoplus_{l=0}^{\infty} \phi_l(\mathfrak{R}_l).$$

Since $\phi_l(\mathfrak{R}_l) \subseteq \phi_l(\mathfrak{M}) \subseteq M_{n_k}(\mathbb{C})$, this decomposition of $\mathfrak{M}$ is an injection $\mathfrak{M} \hookrightarrow \bigoplus_{k=0}^{\infty} M_{m_k}(\mathbb{C})$. The property $\mathfrak{R}_{l+1} \subseteq \bigcap_{k=0}^{l} \ker \phi_k$ ensures that this map is actually a surjection. $\square$

An immediate corollary of the proposition 1.9 is the following

Corollary 1.11. *A state* σ *of* $\mathrm{Comm}(D)$ *is pure if and only if there is a minimal projection* $p \in \mathrm{Comm}(D)$ *such that*

$$\sigma(a) = pap \in p\,\mathrm{Comm}(D)p \simeq \mathbb{C}. \tag{3}$$

The isomorphism $p\,\mathrm{Comm}(D)p \simeq \mathbb{C}$ is actually a caracterisation of the *minimal projection* p. Given a minimal projection p in $\mathrm{Comm}(D)$, we call σ_p the pure state defined by the formula (3).

Elaborating on the previous corollary, we can define the *extremal restriction classes* as the restriction classes coming from pure states of $\mathrm{Comm}(D)$. These classes have a simple description:

Proposition 1.12. *Let* $[\sigma]^r$ *be the extremal restriction class associated to the pure state* σ_p*. This restriction class is homeomorphic, in the weak-$*$-topology sense, to the pure states* $\mathscr{S}(p\mathscr{A}p)$ *of* $p\mathscr{A}p$*.*

Proof. The expression of σ_p provides us with $\sigma_p(1-p) = 0$. If we take a state σ of $\mathscr{A}$ in the restriction class of σ_p, we therefore get $\sigma(1-p) = 0$. Any element $a \in \mathscr{A}$ can be writtten as

$$a = a(1-p) + (1-p)ap + pap.$$

The Cauchy-Schwarz inequality ensures that $\sigma\big(a(1-p)\big) = 0 = \sigma\big((1-p)ap\big)$, which implies $\sigma(a) = \sigma(pap)$.

Thus, any σ in the restriction class of σ_p defines a state $\overline{\sigma}$ on $p\mathscr{A}p$ and conversely if we start with a state $\overline{\sigma}$ of $p\mathscr{A}p$, we can define a state σ on $\mathscr{A}$ by

$$\sigma(a) = \overline{\sigma}(pap).$$

Notice that it follows from the explicit expressions that the operations $\sigma \rightsquigarrow \overline{\sigma}$ and $\overline{\sigma} \rightsquigarrow \sigma$ are continuous in the sense of weak-$*$-topology.

Finally, we use the characterisation of pure states as extremal states. In particular, if $\sigma = t\sigma_1 + (1-t)\sigma_2$ where $t \in (0,1)$ and σ_i are states of $\mathscr{A}$, then $\sigma_p = t\sigma_{1|\,\mathrm{Comm}(D)} + (1-t)\sigma_{2|\,\mathrm{Comm}(D)}$ and since σ_p is pure, $\sigma_{1|\,\mathrm{Comm}(D)} = \sigma_{2|\,\mathrm{Comm}(D)} = \sigma_p$. Hence, σ is pure if and only if $\sigma(p \bullet p) \in \mathscr{S}(p\mathscr{A}p)$ is pure. $\square$

2 Scalar Perturbation Type Spectral Triple

2.1 Definition

Let M be a compact, connected spin manifold without boundaries. Let D_E be its canonical Dirac operator acting on the Hilbert space $\mathscr{H}_0$ of square integrable spinors. The *scalar perturbation type spectral triples* are the spectral triples $(C^\infty(M) \otimes M_n(\mathbb{C}), \mathscr{H}_0 \otimes M_n(\mathbb{C}), D)$ where D is a *scalar perturbation type* Dirac operator, *i.e.* it satisfies

$$[D, a] = [D_E \otimes \mathrm{Id}, a] + [(\Gamma_E \otimes \mathrm{Id})H, a],$$

for some smooth self-adjoint matrix-valued function H on M.

For this class of spectral triples, we will fully describe the bounded components and the *set* of all bounded components. Our description will depend on:

Theorem. Given a spectral triple $(C^\infty(M) \otimes M_n(\mathbb{C}), \mathscr{H}, D)$ of scalar perturbation type, we can find $C > 0$ and $\mathscr{A}_1$ such that $\mathscr{A}_1 \oplus \mathrm{Comm}(D) = \mathscr{A}$ and:

$$\forall a \in \mathscr{A}_1,\ \|a\| \leqslant C\Big\|[D, a]\Big\|,$$

where we used the notation $\mathrm{Comm}(D)$ for $\{a \in \mathscr{A} : [D,a] = 0\}$. To establish our theorem, we need to distinguish according to the dimension N of the manifold M. In a first part, we will show a preliminary result for $N = 1$, then show the same result for $N > 1$, before we can merge both cases into the theorem.

2.2 Elementary Cases

Let us first do some observations on the triple $(\mathscr{A}, \mathscr{H}, D)$ with $\mathscr{A} = C^\infty(M) \otimes M_n(\mathbb{C})$, $\mathscr{H} = \mathscr{H}_0 \otimes M_n(\mathbb{C})$, $D = D_E \otimes \mathrm{Id}_{M_n(\mathbb{C})}$. In this context, the smooth functions on M are seen as sections of the Clifford bundle $\mathbb{C}l(M)$ over M. The sections of this bundle act on $\mathscr{H}_0$. Therefore, if $f \in C^\infty$, we can consider $[D_E, f]$ as a commutator in $B(\mathscr{H}_0)$.

It is well known (see [17], proposition 9.11, p.387) that $[D_E, f] = -idf$, where df is seen as a section of the Clifford bundle $\mathbb{C}l(M)$ acting on $\mathscr{H}_0$. It is obvious in this case that $\mathrm{Comm}(D_E \otimes \mathrm{Id})$ is reduced to the subspace $1 \otimes M_n(\mathbb{C}) \subseteq \mathscr{H}_0 \otimes M_n(\mathbb{C})$. If we choose a point $x_0 \in M$, the set $\mathscr{A}_1 = \{a \in \mathscr{A} : a(x_0) = 0\}$ is a complement of $\mathrm{Comm}(D_E \otimes \mathrm{Id})$ in $\mathscr{A}$.

For any Riemannian manifold M_0 and any two points $x, y \in M_0$, we will use the notation $d_0(x,y)$ for the inferior bound of the length of smooth paths going from x to y. If a is the matrix (a_{ij}), we will refer to the matrix (da_{ij}) as da. We are now ready to prove the following lemma :

Lemma 2.1. *Let M_0 be a bounded connected Riemannian manifold and $x_0 \in M_0$. Let $C_0 = \sup_{x \in M_0}\{d(x, x_0)\}$. For any $a \in C^\infty(M_0) \otimes M_n(\mathbb{C})$ satisfying $a(x_0) = 0$ we have:*

$$\|a\| \leqslant C_0 \Big\| da \Big\|.$$

Remark 2.2. In the case of a spin manifold, we have $\|da\| = \big\|[D_E \otimes \mathrm{Id}, a]\big\|$ and the functions that vanish at x_0 are a complement of $\mathrm{Comm}(D) = 1 \otimes M_n(\mathbb{C})$, thus the theorem is proved.

Proof. Let x be a point of M_0. According to the definition of C_0, for any $\varepsilon > 0$, we can find a smooth path from x_0 to x, with length less than $C_0 + \varepsilon$. If $c(t)$ is such a path, we have:

$$a(x) = a(x) - a(x_0) = \int_{t=0}^{1} da(c(t)).c'(t)$$

We see that the right-hand side is bounded by $(C_0 + \varepsilon) \sup_{u \in M} \|da(u)\|$. Taking ε to 0, we get the estimation of the lemma. $\square$

2.3 The case of S^1

We now establish our result in the simple case of $N = 1$. The only compact connected boundaryless manifold of dimension 1 is the circle S^1. In this case, the Dirac operator $D_E = -i\frac{d}{d\theta}$, and there is no difference between a scalar perturbation and a general perturbation of the Dirac operator.

Lemma 2.3. *For any perturbation D of the Dirac operator $D_E \otimes \mathrm{Id}_{M_n(\mathbb{C})}$ on $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$, if we set $\mathscr{A}_0 = \{a \in \mathscr{A} : a(\pi) = 0\}$ then $\mathrm{Comm}(D) \cap \mathscr{A}_0 = \{0\}$ and for any $a \in \mathscr{A}_0$, the following holds:*

$$\|a\| \leqslant \pi \big\|[D, a]\big\|,$$

where the norm is that of $B(\mathscr{H})$.

Proof. The commutator is $-ia'(\theta) + [H, a](\theta)$. We can find the primitive K of H on $[0, 2\pi]$ with $K(0) = \mathrm{Id}$ and then write :

$$[D, a](\theta) = \left(e^{-iK}\left[-i\frac{d}{d\theta} \otimes \mathrm{Id}, e^{iK}ae^{-iK}\right]e^{iK}\right)(\theta)$$

for any $\theta \in [0, 2\pi]$. K being self-adjoint, e^{iK} is unitary. Clearly, $[D, a] = 0$ implies $e^{iK}ae^{-iK} = L_0$ where $L_0 \in 1 \otimes M_n(\mathbb{C})$, that is $a = e^{-iK}L_0e^{iK}$. It is now obvious that $\mathrm{Comm}(D) \cap \mathscr{A}_0 = \{0\}$.

Let us now prove the estimate. e^{iK} being unitary,

$$\|[D, a]\| = \left\|\left[-i\frac{d}{d\theta} \otimes \mathrm{Id}, e^{iK}ae^{-iK}\right]\right\|.$$

Now, if we take $x_0 = \pi$ and $a(x_0) = 0$, we have $\left(e^{iK}ae^{-iK}\right)(\pi) = 0$, and we can use the lemma 2.1 with $C_0 = \pi$:

$$\|a\| = \|e^{iK}ae^{-iK}\| \leqslant C_0\|[D_E \otimes \mathrm{Id}, e^{iK}ae^{-iK}]\| = \pi\|[D, a]\|.$$

$\square$

2.4 Generic Case

We will now treat the generic case.

The spectral triple over the manifold M can either be *odd* or *even*, according to the parity of N, the dimension of M. In the even case, a self-adjoint unitary operator Γ_E on $\mathscr{H}_0$ is available. It commutes with $\mathscr{A}$ and anti-commutes with D_E.

In the odd case, this operator Γ_E is not available, so we will let $\Gamma_E = \mathrm{Id}$ in this case.

Lemma 2.4. *If $N > 1$, we can find a constant C_2 such that for any $a \in \mathscr{A}$:*

$$\|[D_E \otimes \mathrm{Id}, a]\| \leqslant C_2 \|[D, a]\|$$

Remark 2.5. In particular, if $a \in \mathrm{Comm}(D)$, then $\|[D_E \otimes \mathrm{Id}, a]\| = 0$ and $a \in 1 \otimes M_n(\mathbb{C})$.

Proof. We shall distinguish two cases, according to the parity of N. These cases will not require the same arguments. However, in both cases we will exhibit a continuous Banach space projection P such that

$$P([D, a]) = [D_E \otimes \mathrm{Id}, a].$$

In the even case, we can consider an involution on $B(\mathscr{H})$ defined by $S(T) = (\Gamma_E \otimes \mathrm{Id})T(\Gamma_E \otimes \mathrm{Id})$. This involution is clearly continuous, thus the associated projections $\frac{1}{2}(\mathrm{Id}_{B(\mathscr{H})} \pm S)$ are continuous.

Now, mind that for all $f \in C^\infty(M)$, $\Gamma_E f = f \Gamma_E$ and that $D_E \Gamma_E = -\Gamma_E D_E$, hence if we take $[D, a] = [D_E \otimes \mathrm{Id}, a] + [(\Gamma_E \otimes \mathrm{Id})H, a]$, we see that

$$\frac{1}{2}(\mathrm{Id}_{B(\mathscr{H})} - S)([D, a]) = [D_E \otimes \mathrm{Id}, a]$$

and the proposition is proved.

In the odd case, by decomposing a according to the canonical basis (e_{ij}) of $M_n(\mathbb{C})$, we can reduce the problem to $C^\infty(M)$. Taking the notations of [17], p.373, $[D_E, f] + f'$ acts as $-ic(df\gamma) + c(f')$ where c is the faithful representation of $\mathbb{C}l^+(M)$ on the spinors and γ is the *chirality element* defined p.371. We have ([17], p.370) $\mathbb{C}l^+(M) \simeq \bigoplus_k \Lambda^{2k} T^*M$ (as a vector bundle) and we can split $\mathbb{C}l^+(M)$ into $\mathcal{S} \simeq \bigoplus_{n>0} \Lambda^{2n} T^*M$ and $\mathcal{T} \simeq \Lambda^0 T^*M$, the trivial complex bundle. We want to use this splitting $\mathbb{C}l^+(M) = \mathcal{S} \oplus \mathcal{T}$ to justify our projection.

It is clear that f' is a section of $\mathcal{T}$, therefore we just have to check that $df\gamma$ is a section of $\mathcal{S}$. We will simply check that for any $x \in M$, $df(x)$ is an element of the fiber $\mathcal{S}_x$.

If $df(x) = 0$, it is obvious. Therefore, we must take $x \in M$ such that $df(x) \neq 0$. On a small enough neighbourhood of x, using the flow-box theorem we can find a local orthonormal basis of 1-forms $(\theta_1, \cdots, \theta_N)$ such that $df = \alpha\theta_1$, where α is a smooth function. Define m by $2m+1 = N$, then $df(x)\gamma(x) = (-i)^m \alpha(x)\theta_1(x)\theta_1(x)\cdots\theta_N(x) = (-i)^m \alpha(x)(\theta_1, \theta_1)(x)\theta_2(x)\cdots\theta_N(x)$. This sum contains $N-1$ terms, and therefore belongs to $\mathcal{S}_x$ as soon as $N > 1$.

Since $\mathscr{S}$ is a closed complemented space of $\mathbb{C}l^+(M)$, there is a continuous projection from $\mathbb{C}l^+(M)$ to $\mathscr{S}$, as discussed in [5], p.16. □

Remark 2.6. In fact, we can use the above proof for both cases, if we assume that Γ_E is the chiral element γ, but the even case is so much simpler that we preferred including both proofs.

As a corollary, we get:

Corollary 2.7. *For any perturbation D of the Dirac operator $D_E \otimes \mathrm{Id}_{M_n(\mathbb{C})}$ on M of dimension at least two, if we take a point $x_0 \in M$ and set $\mathscr{A}_0 = \{a \in \mathscr{A} : a(\pi) = 0\}$ then $\mathrm{Comm}(D) \cap \mathscr{A}_0 = \{0\}$ and for any $a \in \mathscr{A}_0$, the following holds:*

$$\|a\| \leqslant C_1 \|[D, a]\|,$$

where the norm is that of $B(\mathscr{H})$.

Proof. Using the remark after lemma 2.4, if $a \in \mathrm{Comm}(D)$, then $a \in 1 \otimes M_n(\mathbb{C})$. It is consequently clear that $\mathscr{A}_0 \cap \mathrm{Comm}(D) = \{0\}$.

The estimate on the commutator with $C_1 = C_0 C_2$ follows immediately from the lemmas 2.1 and 2.4. □

We can now state our theorem:

Theorem 2.8. *For a scalar perturbation type spectral triple $(C^\infty(M) \otimes M_n(\mathbb{C}), \mathscr{H}, D)$, we can find $C > 0$ and $\mathscr{A}_1$ such that $\mathscr{A}_1 \oplus \mathrm{Comm}(D) = \mathscr{A}$ and:*

$$\forall a \in \mathscr{A}_1, \qquad \|a\| \leqslant C \|[D, a]\|. \tag{4}$$

Proof. From lemma 2.3 or corollary 2.7 (depending on the dimension), we see that:

$$\forall a \in \mathscr{A}_0, \qquad \|a\| \leqslant C_1 \|[D, a]\|. \tag{5}$$

The set $\mathscr{A}_0$ has finite codimension and $\mathscr{A}_0 \cap \mathrm{Comm}(D) = \{0\}$, hence we can find a finite dimensional space V such that $\mathscr{A}_0 \oplus V \oplus \mathrm{Comm}(D) = \mathscr{A}$. We set $\mathscr{A}_1 = \mathscr{A}_0 \oplus V$.

It is obvious that $\|[D, a]\| = \|a\|_1$ is a norm on $\mathscr{A}_1$, and we can take $A_1 = \overline{\mathscr{A}_1}^1$, the completion of $\mathscr{A}_1$ for the norm $\| \, \|_1$.

We want to prove that $A_1 = \overline{\mathscr{A}_0}^1 \oplus V$. On the one hand, $\mathscr{A}_0 \subseteq A_1$ and A_1 is closed for $\| \, \|_1$, so $\overline{\mathscr{A}_0}^1 \subseteq A_1$; on the other hand, $V \subseteq A_1$ and V is obviously closed.

We just have to check that $\overline{\mathscr{A}_0}^1 \cap V = \{0\}$. Set $\Psi(a) = a(x_0)$. This matrix-valued linear form is clearly C^*-norm continuous on $\mathscr{A}_0$. Considering the equation (5), we see that Ψ is also $\| \, \|_1$-continuous. But $\Psi = 0$ on

$\mathscr{A}_0$ and so $\overline{\mathscr{A}_0}^1 \subseteq \ker \Psi$. As $V \cap \ker \Psi = \{0\}$, we see that $\overline{\mathscr{A}_0}^1 \oplus V = A_1$.

We have two closed complemented spaces in a Banach space, and the projection P on V is well defined and continuous, *i.e.* $\|P(a)\|_1 \leqslant C_4 \|a\|_1$. In fact, by taking an isometric embedding of $A_1/\overline{\mathscr{A}_0}^1$ in A_1, we can choose V_0 such that the projection on V_0 along $\overline{\mathscr{A}_0}^1$ has norm 1.

V_0 being finite dimensional, we can find a constant C_3 such that: $\|P(a)\| \leqslant C_3 \|P(a)\|_1$. Putting it all together:

$$\begin{aligned}\|a\| \leqslant \|a - P(a)\| + \|P(a)\| \leqslant & C_1 \|a\|_1 + C_1 \|P(a)\|_1 + C_3 \|P(a)\|_1 \\ \leqslant & (2C_1 + C_3) \|a\|_1.\end{aligned}$$

□

The equation (4) associated with the property that for any $a \in \mathscr{A}_1$, $\|a_1\|^{\sim} \leqslant \|a_1\|$ shows that the hypotheses of lemma 1.4 are fulfilled. Hence,

Corollary 2.9. *For a scalar perturbation type spectral triple over $C^\infty(M) \otimes M_n(\mathbb{C})$, the restriction classes coincide with the bounded components.*

3 Bounded Components of Scalar Perturbations

3.1 Description of the Bounded Components

If we only consider *pure* states, we can give a more precise description of the "pure states bounded components".

First, notice that $\mathrm{Comm}(D)$ is a sub-*-algebra of $M_n(\mathbb{C})$, *i.e.* it is a finite dimensional C^* algebra, so we can write:

$$\mathrm{Comm}(D) = \bigoplus_i B_i$$

where $B_i = M_{n_i}(\mathbb{C})$. Each B_i is represented in $M_n(\mathbb{C})$ with multiplicity m_i and $1_{\mathscr{A}} \in \mathrm{Comm}(D)$, thus $\sum_i n_i m_i = n$.

We can now characterise the bounded components of pure states:

Proposition 3.1. *Let σ be a pure state of A. We can find a system of $\alpha_i \in [0,1]$ and σ_i, pure states of B_i such that $\sum_i \alpha_i = 1$ and $\sigma_{|B_i} = \alpha_i \sigma_i$.*

Conversely, given $\alpha_i \in [0,1]$ with $\sum_i \alpha_i = 1$ and $(\sigma_i)_i$, pure states of B_i, we can always find a pure state of A such that $\sigma_{|B_i} = \alpha_i \sigma_i$.

Moreover, if m is the number of nonzero α_i, the set of the pure states σ which extend (α_i, σ_i) is homeomorphic – in the sense of the weak $*$ topology

– *to* $M \times T^{m-1} \prod_i P(\mathbb{C}^{m_i})$ *where* $P(\mathbb{C}^{n_i}) = \mathbb{C}^{n_i}/\mathbb{C}^*$ *is the projective space over* $\mathbb{C}^{n_i}$*.*

Proof. Bear in mind that defining a pure state of $M_n(\mathbb{C})$ is nothing but taking a rank 1 projection of $\mathbb{C}^n$. Given a rank 1 projector $\overline{p_i}$ of B_i, we know its image in $M_n(\mathbb{C})$ has rank m_i and we have to choose a rank 1 subprojection of this image.

This proposition follows from elementary consideration about rank one matrices. □

Remark 3.2. Notice that in our case, what we call "bounded components" are in fact connected components of the state of pure states, in the sense of the topology induced by the distance d.

Definition 3.3. If $[\sigma]$ is a bounded component, we call *amplitude* of $[\sigma]$ the number m of nonzero α_i.

There is a partition of the bounded components according to their amplitude. Notice that a bounded component of amplitude $m = 1$ is homeomorphic to M.

In the other cases, here is a description of the geometry of a piece of the partition:

Proposition 3.4. *Take* $m \neq 1$*. Let* S *be the set of the bounded components of amplitude* m *such that the* m *first* α_i *are nonzero. There is a bijection from* S *onto* $\Sigma^m \times \prod_{i=1}^m P(\mathbb{C}^{n_i})$ *where* Σ^m *is the interior of the* m *dimensional simplex.*

Proof. Indeed, the choice of a bounded component of S is just the choice of m nonzero α_i and of m directions, each direction belonging to a $\mathbb{C}^{n_i}$. Notice that none of the m first α_i can be equal to 1, otherwise the others should be 0. □

3.2 Topology of the Bounded Components

In the following, we only consider manifolds of dimension strictly more than 1.

Proposition 3.5. *For a scalar type perturbation spectral triple, the metric and weak-$*$ topologies agree on restriction classes.*

Proof. We want to apply proposition 1.8. First notice that equation (4) implies that the image of $\mathcal{L}_1$ is included in the image of $E = \{a \in \mathscr{A}_1 : \|a\| \leqslant$

C and $\|[D,a]\| \leqslant 1\}$. Hence, E is a set of derivable functions on a compact space which are uniformly bounded.

Moreover, Lemma 2.4 proves that the functions in E have uniformly bounded derivatives. Hence, the family E is equicontinuous and we can apply Ascoli's theorem to prove that E is totally bounded. Consequently the image of E is totally bounded, and the hypotheses of proposition 1.8 are satisfied. □

3.3 Possible Decompositions of $\mathrm{Comm}(D)$

Let us take a constant H, which can therefore be decomposed by blocks into:

$$H = \begin{pmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_n I_{n_k} \end{pmatrix},$$

where the λ_i are couplewise distinct. Using the remark after lemma 2.4, $\mathrm{Comm}(D) \subseteq 1 \otimes M_n(\mathbb{C})$. If a is in $\mathrm{Comm}(D)$, it must satisfy $[H,a]=0$, *i.e.* a must be written:

$$a = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_k \end{pmatrix},$$

where the M_i belong to $M_{n_i}(\mathbb{C})$. It is easy to check that any such element is in $\mathrm{Comm}(D)$. Thus, we see that any decomposition

$$\mathrm{Comm}(D) = \bigoplus_i B_i$$

where $B_i = M_{n_i}(\mathbb{C})$ and $\sum_i n_i = n$, can actually appear.

References

[1] I. Androulidakis and G. Skandalis. The holonomy groupoid of a singular foliation. *Journal für die Reine und Angewandte Mathematik. [Crelle's Journal]*, 626:1–37, 2009.

[2] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[3] E. Cagnache, F. D'Andrea, P. Martinetti, and J. Wallet. The spectral distance on the Moyal plane. *0912.0906*, Dec. 2009.

[4] E. Cagnache and J.-C. Wallet. Spectral Distances: Results for Moyal Plane and Noncommutative Torus. *Symmetry, Integrability and Geometry: Methods and Applications*, 2010.

[5] N. L. Carothers. *A Short Course on Banach Space Theory*. Cambridge University Press, Dec. 2004.

[6] A. Connes. Gravity coupled with matter and the foundation of noncommutative geometry. *Commun. Math. Phys.*, 182:155–176, 1996.

[7] A. Connes. On the spectral charaterization of manifolds. preprint, http://www.alainconnes.org/docs/reconstructionshort.pdf, May 2007.

[8] M. Hilsum. Bordism invariance in KK-theory. *Mathematica Scandinavica*, 107(1):73–89, 2010.

[9] B. Iochum, T. Krajewski, and P. Martinetti. Distances in finite spaces from noncommutative geometry. *Journal of Geometry and Physics*, 37(1-2):100–125, 2001.

[10] P. Martinetti. Carnot-Carathéodory metric and gauge fluctuation in noncommutative geometry. *Communications in Mathematical Physics*, 265(3):585–616, 2006.

[11] P. Martinetti. Spectral distance on the circle. *Journal of Functional Analysis*, 255(7):1575–1612, 2008. Journal of Functional Analysis 255 (2008), pp. 1575-1612.

[12] P. Martinetti and R. Wulkenhaar. Discrete Kaluza-Klein from scalar fluctuations in noncommutative geometry. *Journal of Mathematical Physics*, 43(1):182–204, 2002.

[13] B. Pavlovic. Defining metric spaces via operator from unital C*-Algebras. *Pacific Journal of Mathematics*, 186(2):285–313, 1998.

[14] G. K. Pedersen. *Analysis Now*, volume 118 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.

[15] M. A. Rieffel. Metrics on states from actions of compact groups. *Documenta Mathematica*, 3:215–229 (electronic), 1998.

[16] M. A. Rieffel. Metrics on state spaces. *Documenta Mathematica*, 4:559–600 (electronic), 1999.

[17] J. C. Varilly, H. Figueroa, and J. M. Gracia-Bondia. *Elements of Non-commutative Geometry.* Birkhäuser Boston, 1 edition, Oct. 2000.